Japan Bilingual Publishing Co.

Mathematical Methods in Data Science

https://ojs.bilpub.com/index.php/mmds

ARTICLE

Computational Algorithm of Fuzzy Arithmetic Based on the Principle of Maximum Entropy

Sergey Sveshnikov * **, *Victor Bocharnikov***

Center for Military-Strategic Studies, National Defense University of Ukraine, Kyiv 03049, Ukraine

ABSTRACT

Fuzzy numbers are a powerful tool for working with numerical uncertainty, but they remain little used in solving practical problems in finance and economics. The main problem lies in the significant extension of the support of a fuzzy number when a large number of arithmetic transformations are performed. For example, when calculating profit, the minimum and maximum support values can differ by several times. In such cases, making an investment decision is extremely difficult. Therefore, we developed a new algorithm that generates the result of an arithmetic operation as a projection of a fuzzy binary relation, selected based on the principle of maximum entropy. The proposed algorithm generates a matrix of a fuzzy binary relation in accordance with Zadeh's extension principle. The principle of maximum entropy implies choosing the relation projection (row or column) that contains the cell with the highest degree of membership in the matrix and that has the highest entropy. Compared to standard fuzzy arithmetic, the proposed algorithm provides a much smaller extension of the domain of the resulting fuzzy number than existing methods. In addition, the proposed algorithm works with operands whose support crosses zero. The algorithm is implemented as a Microsoft Excel add-in, which readers can download free of charge and use for applied calculations. We also described an example of calculating the profit and currency risk of options using the proposed algorithm.

Keywords: Algorithm; Fuzzy Set; Fuzzy Number; Arithmetic Operation

*CORRESPONDING AUTHOR:

Sergey Sveshnikov, Center for Military-Strategic Studies, National Defense University of Ukraine, Kyiv 03049, Ukraine; Email: sv367@ukr.net

ARTICLE INFO

Received: 20 August 2025 | Revised: 16 October 2025 | Accepted: 23 October 2025 | Published Online: 30 October 2025

DOI: https://doi.org/10.55121/mmds.v1i1.1245

CITATION

Sveshnikov, S., Bocharnikov, V., 2025. Computational Algorithm of Fuzzy Arithmetic Based on the Principle of Maximum Entropy. Mathematical Methods in Data Science. 1(1): 1–19. DOI: https://doi.org/10.55121/mmds.v1i1.1245

1. Introduction

The solution of applied problems in different areas is often based on the use of numerical quantities, the values of which can be determined only approximately. The uncertainty of these quantities can have a different nature and can be caused by incomplete data, inaccuracy of the linguistic description, as well as other reasons.

Researchers often describe uncertain numerical quantities using fuzzy numbers, which are fuzzy sets defined on a set of real numbers. The need to use fuzzy numbers is especially relevant in economic problems, where numerical quantities are subjected to multiple arithmetic transformations.

Fuzzy numbers and fuzzy arithmetic were introduced by Zadeh and improved by Dubois, Prade[1, 2], as well as other researchers, including Yager[3], Mizumoto and Tanaka[4].

In the general case, as a fuzzy number A is called a fuzzy set with a support defined on the set of real numbers:

$$A = \{(x, y) \in \mathbb{R} \times [0, 1] : y = \mu_A(x)\}, \quad (1)$$

where $\mu_A(x)$ is the membership function of A.

Researchers also use fuzzy intervals. Fuzzy interval is called the normal fuzzy set defined on the set of real numbers[5], for which the following conditions are met:
$\exists x \in \mathbb{R}, \mu_A(x) = 1$;
α-cut of A is closed interval ${}^{\alpha}A = \{x \in \mathbb{R}\ |\mu_A(x) \geq \alpha \in (0, 1].\}$;
support of A is defined as: $\text{SuppA} = \{x \in \mathbb{R}\ |\mu_A(x) > 0\}$.

The membership function of the fuzzy interval is expressed as follows:

$$\mu_A(x) = \begin{cases} f_A(x) & \text{when } x \in [a, b), \\ 1 & \text{when } x \in [b, c], \\ g_A(x) & \text{when } x \in (c, d] \\ 0 & \text{otherwise}, \end{cases} \quad (2)$$

where $a, b, c, d \in \mathbb{R}$, $a \leq b \leq c \leq d$, $f_A(\cdot)$ is a real-valued function that is increasing and right-continuous, and $g_A(\cdot)$ is a real-valued function that is decreasing and left-continuous. Various versions of these functions are presented in the paper[6].

A fuzzy interval with a one-mode membership function is called a fuzzy number: $\mu_A(x) = 1$ for exactly one $x \in \mathbb{R}$.

The implementation of any arithmetic operation $\bigotimes$ on two fuzzy numbers A and B is based on the use of the Zadeh's extension principle:

$$\mu_C(z) = \sup_{z = x \bigotimes y} \min\{\mu_A(x),\ \mu_B(y)\}, x, y, z \in \mathbb{R}, \quad (3)$$

where $\mu_C(z)$ is the membership function of the resulting fuzzy number $C = A \bigotimes B$.

It is assumed that arithmetic operations with fuzzy numbers should retain the properties of standard arithmetic, in particular: commutativity, associativity, distributivity, properties of zero, unit and multiplication.

In applied problems, fuzzy quantities cannot always be adequately described by fuzzy numbers with a standard form of the membership function, since the nature of the uncertainty for different quantities, as a rule, is different. Based on this, the main requirements for the algorithm for performing arithmetic operations with fuzzy numbers are as follows:

- The properties of arithmetic operations with fuzzy numbers should not contradict the properties of standard arithmetic;
- The form of the membership function of the result must correctly take into account the form of the membership functions of the operands of the arithmetic operation;
- The result of an arithmetic operation should have as narrow support as possible in order to provide the unambiguity necessary to prevent erroneous solutions;
- The computational complexity of algorithms should be minimized as much as possible in order to ensure the efficiency of calculations.

The most common algorithms are based on the Zadeh's extension principle and implement arithmetic operations with fuzzy numbers that are approximated by several points on the set of real numbers. In particular, three points are used to describe triangular fuzzy numbers; four points are used to describe trapezoidal fuzzy numbers, and so on. These algorithms are easy to implement, but there are three problematic questions here. Firstly, the approximation of fuzzy quantities leads to errors that may be unacceptable in practice. Secondly, the shape of the resulting fuzzy number is sometimes not related to the shape of the operands of an arithmetic operation, that is, in some cases, it is difficult to explain the change in the nature of the result. Thirdly, the use of these algorithms leads to a significant increase in the support of the resulting fuzzy number in the case of multiple transformations. Sometimes, the right border of the support of a fuzzy number can be several times larger than the left one. In such conditions, it is very difficult to make an adequate decision.

Therefore, we have developed an algorithm that provides the fulfilment of arithmetic operations with fuzzy numbers of arbitrary form and increases the support of the result to a lesser extent than the known algorithms. The novelty of our proposed method lies in the discretization of fuzzy numbers along the abscissa axis (on the set of real numbers), the calculation of a fuzzy binary relation in accordance with the Zadeh's extension principle and the selection of a projection of this relation that contains the maximum membership and has the maximum entropy.

2. Materials and Methods

2.1. Related Researches

Researchers often use fuzzy numbers with triangular membership functions[7]: $f_A(x) = \frac{x-a}{b-a}$, $g_A(x) = \frac{d-x}{d-c}$, $b = c$ in Equation (2).

In this case, the fulfilment of arithmetic operations is simplified, and the result is calculated using simple algebraic expressions. However, if the membership functions of the operands are represented by asymmetric triangles, the question arises about the shape of the membership function of the result, as shown in **Figure 1**. This example shows that the membership function of the result can have a different shape than the membership functions of both operands. This fact cannot be rationally explained.

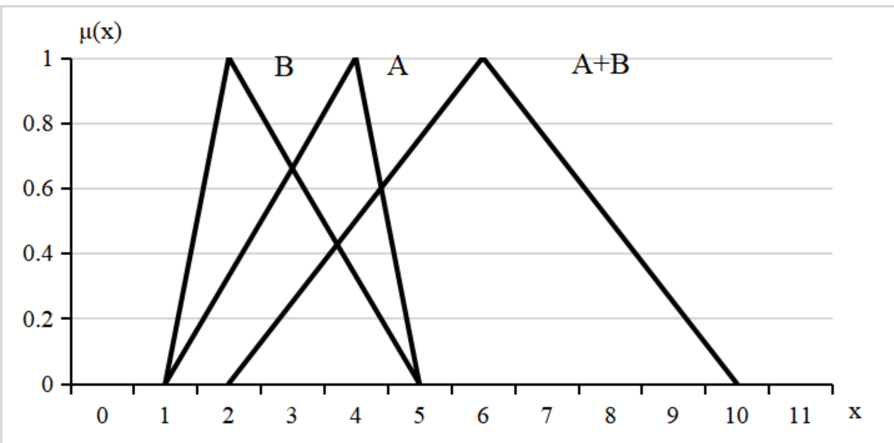


Figure 1. Changing the shape of the membership function when summing two triangular fuzzy numbers A and B.

Bede researched the operation of multiplying fuzzy numbers and confirmed that the shape of a fuzzy number can change[8].

From the other side, the computational complexity of arithmetic operations with triangular fuzzy numbers is minimal, but in practice, the approximation errors of fuzzy quantities are quite big. Representation of fuzzy numbers by trapezoids[9], pentagons[10], piecewise linear functions or nonlinear membership functions reduces these errors[11–14], however, the question concerning the shape of the membership function of the result remains. Hanss notes the loss of information in the case of triangular fuzzy numbers and suggests using discretized fuzzy numbers to solve engineering problems[15]. In papers[16, 17], the authors research an RDM method for performing arithmetic operations. This method uses a representation of fuzzy numbers as a set of level slices. However, the extension of the support is also preserved here, since the method uses the Zadeh's extension principle. The Hukuhara generalized difference method[18, 19] can be used to perform arithmetic operations. However, it requires additional research, as in some cases it completely destroys uncertainty. For example, this is the case of the difference between the two symmetric numbers shown in **Figure 1**. Furthermore, within the framework of RDM arithmetic, the domain of the result using the Hukuhara method is identical to the domain of the result using standard arithmetic.

With regard to arithmetic operations with fuzzy numbers, research is aimed mainly at ensuring that the properties of standard arithmetic are fulfilled. Dubois and Prade identified these properties [2]. In particular, in fuzzy arithmetic, the definition of a fuzzy zero for the summation operation is a problem question, and the definition of a fuzzy unit is a problem question for multiplication and division. These problems are discussed in detail in a study by Mares [20], which suggests replacing in properties a strict equality relation with a weaker equivalence relation. Other studies, for example Piasecki and Łyczkowska-Hanćkowiak [21], to solve the problems of fuzzy arithmetic, in description of fuzzy numbers suggest using additional elements, in particular, their orientation.

However, from the point of view of applied problems, the main negative property of most known approaches to the implementation of arithmetic operations on fuzzy numbers is a sharp increase of the support in the case of many transformations [22, 23]. This property becomes an important drawback in applied problems in which the calculation algorithm repeatedly transforms the initial fuzzy quantities.

Consider a simple example. **Figure 2** illustrates the summation of two fuzzy numbers that have the support power $|\text{Supp}(A)| = |\text{Supp}(B)| = 4$. The result of the summation has the support power $|\text{Supp}(A + B)| = 8$. If we sum several times the number B with the result, then the support power will increase by 4 each time. After five operations, the support power of the result will increase to $|\text{Supp}(A + B + B + B + B + B)| = 24$, that is, six times compared to the initial fuzzy number (see **Figure 2**). The result uncertainty also increases dramatically. In particular, for this example, the cardinality of the fuzzy number will increase from $\text{Card}(A) = 2$ to $\text{Card}(A + B + B + B + B + B) = 12$.

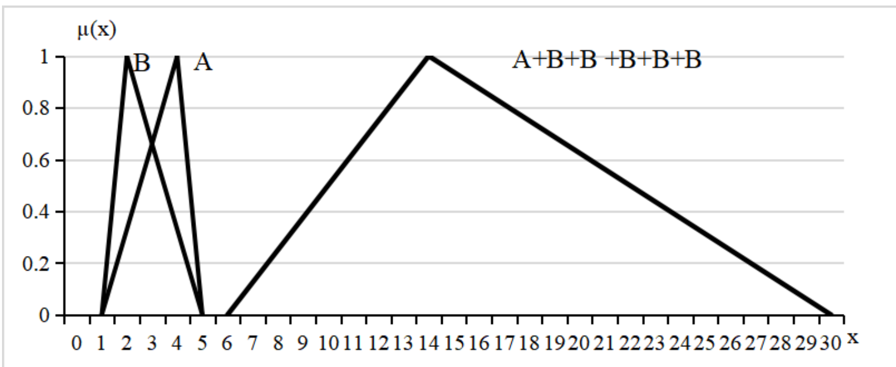


Figure 2. Illustration of multiple summation of fuzzy numbers.

It is obvious that a wide variation of the possible values of the fuzzy quantity makes calculations with fuzzy numbers unsuitable for decision-making in applied problems. If we assume that in the above example, the result describes the predicted profit of a business project, then the range from 6 to 30 monetary units is not suitable to conclude that the project is expedient, since this estimate is almost equivalent to complete uncertainty.

To limit the increase in the support of a fuzzy number, Klir [5] suggested using additional information that standard fuzzy arithmetic [1] does not take into account. He presented this information in the form of "necessary constraints" that arise from the meaning of operands. For example, an equality constraint should be taken into account when both operands are states of the same linguistic variable whose base variable cannot have two different values at the same time. Klir suggested adding to the standard fuzzy arithmetic the constraints in the form of a relation defined on the Cartesian product of two fuzzy intervals. Then the standard Equation (3) is written as follows:

$$\mu_C(z) = \sup_{z = x \otimes y} \min\{\mu_A(x), \mu_B(y), R(x, y)\}, \quad x, y, z \in \mathbb{R}, \tag{4}$$

where R is a crisp or fuzzy binary relation on the set ${}^{\alpha}A \times {}^{\alpha}B$.

Note that the constraints for arithmetic expressions that contain m fuzzy numbers should be formulated as m-dimensional relations.

Using this approach allows you to limit the increase in the support of fuzzy numbers. But when solving applied problems, difficulties arise. Since the necessary constraints flow from the meaning of the operands, it is difficult to develop a computational algorithm that would be universal for all applied problems. In each case, the set of constraints should be different. Based on this, in the universal computational algorithm, it is possible to realize, perhaps, only one necessary constraint—the equality constraint. However, as noted in the study by Klir [5], the use of the equality constraint does not allow for limiting the result carrier in the case of summation.

Thus, many of the approaches mentioned above strive to strictly fulfill the properties of arithmetic operations and at the same time decrease to computational complexity by representing membership function with the help of various approximations. It is important to accurately describe both expert judgments and their correct transformations. One of the main disadvantages that make it difficult to use the known approaches in practice is a sharp increase in the result support in the case of multiple arithmetic transformations.

Therefore, our task is to build an algorithm that should correctly fulfill arithmetic operations with discretized fuzzy numbers and provide a moderate growth of the support of the resulting fuzzy number, as well as the possibility of a clear explanation of changes in the shape of the membership function.

2.2. Proposed Algorithm

The theoretical foundations of the proposed algorithm are presented in the book [24]. In particular, fuzzy numbers are described using fuzzy measures of possibility, and arithmetic operations are built on the basis of fuzzy-integral H-correspondences that satisfy the properties of standard arithmetic: commutativity, associativity, and distributivity.

For implementation in software, we offer a new computational algorithm of arithmetic operations on fuzzy numbers that have an arbitrary, not necessarily convex membership function. Such fuzzy numbers we will call numerical fuzzy sets (NFS). NFS are discretized along the abscissa axis and are represented by pairs in Equation (5). As shown by Dubois and Prade [25], arithmetic operations can be successfully implemented for the case when fuzzy numbers are presented in discretized form.

As shown in the study Hanss [15], discretization can be realized by dividing into equal intervals either the abscissa axis (support of a fuzzy number) or the ordinate axis (membership function). In the first case, a fuzzy number is represented as a set of pairs:

$$A=\{(x_i,\ \mu(x_i)),\ i=\overline{1,n},\ x_{i+1}-x_i=\tfrac{d-a}{n-1},\ x_0=a,\ x_n=d\}, \tag{5}$$

where n is the number of discrete (segments), and d, a are the lower and upper boundaries of the support of the fuzzy number A.

In the second case, a fuzzy number is also represented by many pairs:

$$A=\{(x_i^{\max},\ \alpha_i),(x_i^{\min},\ \alpha_i)\ i=\overline{1,n},\ \alpha_{i+1}-\alpha_i=\tfrac{1}{n-1},\alpha_0=0,\ \alpha_n=1\}, \tag{6}$$

where $x_i^{\min}$ and $x_i^{\max}$ are the lower and upper boundaries of the support of α_i-cut.

Hanss notes that the first case is unsuitable due to possible information losses if the intervals are chosen sufficiently large [15]. However, we can see that the second way also turns out to be unsuitable if the membership function has several extrema. Therefore, to use discretization of the ordinate axis, a fuzzy number must be approximated only by a convex membership function. But by this, we will introduce errors in expert judgment. The best solution is the discretization of the abscissa axis, subject to the Kotelnikov theorem [26]. Such discretization improves the accuracy of describing the shape of FNs, but also increases the computational complexity of the algorithm. Therefore, when choosing the discretization frequency in practical applications, it is necessary to balance the accuracy of the description with the computational complexity. In addition, the choice of discretization frequency will be influenced by the requirements of the subject area where the algorithm is planned to be scaled.

The proposed algorithm is based on representing the result of an arithmetic operation in the form of a fuzzy binary relation of two NFS and choosing from this relation a conditional projection that has a maximal membership function and a maximal degree of uncertainty. Consider this algorithm.

Step 1. Entering discretized NFSs A and B according to Equation (5):

$$\begin{gathered}A=\{(x_i,\ \mu(x_i)),\ i=\overline{1,n},\ x_{i+1}-x_i=\frac{\overline{a}-\underline{a}}{n-1},\\ x_0=\underline{a},\ x_n=\overline{a},\ \mu(x_i)\in[0,1]\},\\ B=\{(y_j,\ \vartheta(y_j)),\ j=\overline{1,n},\ y_{j+1}-y_j=\frac{\overline{b}-\underline{b}}{n-1},\\ y_0=\underline{b},\ y_n=\overline{b},\vartheta(y_j)\in[0,1]\},\end{gathered} \tag{7}$$

where $\underline{a}$ and $\overline{a}$ are lower and upper borders of the support of A, Supp$A = [\underline{a}, \overline{a}]$;
$\underline{b}$ and $\overline{b}$ are lower and upper borders of the support of B, Supp$B = [\underline{b}, \overline{b}]$, $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in \mathbb{R}$.

Step 2. Obtaining a fuzzy binary relation on Cartesian product $D = A \times B$ satisfying the condition:

$$D = \begin{Bmatrix} d_{00} & \dots & d_{0n} \\ \dots & d_{ij} & \dots \\ d_{n0} & \dots & d_{nn} \end{Bmatrix}, \tag{8}$$

where $d_{ij} = \{z_{ij}, \omega_{ij}\}$, $z_{ij} = x_i \bigotimes y_j$, $\omega_{ij} = \min(\mu(x_i), \vartheta(y_j))$,
$\bigotimes$ is designation of an arithmetic operation.

Step 3. Scanning all the elements d_{ij} and composing the set P of index pairs for which the membership function is maximal:

$$\begin{aligned} P = \{(s_k, c_k), k \in \{1, 2, \dots, n^2\} \mid \omega_{s_k c_k} \\ = \max_{i,j=\overline{1,n}} \omega_{ij}\}, \end{aligned} \tag{9}$$

where s_k and c_k are the row and column indices in D.

Step 4. Obtaining of the set E of potential solutions of the arithmetic operation $\bigotimes$.

For each pair of indexes $(s_k, c_k) \in P$ we define two potential solutions, which are the row $D_k^A = \{d_{s_k,j}, j = \overline{1,n}\}$, and the column $D_k^B = \{d_{i,c_k}, i = \overline{1,n}\}$ of the relation D. In this case, $E = \bigcup_{k=\overline{1,|P|}} \{D_k^A \cup D_k^B\}$. Note that for unimodal discretized NFS $|E| = 2$.

Step 5. Calculating the uncertainty degree of each potential solution.

We denote the set E by $E = \{E_l, l = \overline{1, |E|}\}$, where $E_l = \{e_i^l, i = \overline{1,n}\} \subset D$, $e_i^l = \{(z_i^l, \omega_i^l) | z_i^l \in . \mathbb{R}, \omega_i^l \in [0,1]\}$. In this case, by analogy with the entropy (which takes into account both the support and the membership)[27], the uncertainty degree of the discretized NFS is calculated as follows:

$$f(E_l) = \sum_{i=1}^{n-1} \left| z_{i+1}^l - z_i^l \right| \cdot \omega_i^l. \tag{10}$$

Step 6. The choice of the solution R of the arithmetic operation $\bigotimes$ from the set E with a maximal degree uncertainty:

$$R = arg \left\{ \max_{E_l \in E} f(E_l) \right\}. \tag{11}$$

This step implements the well-known principle of maximum entropy. The primary motivation for using this principle is to minimize unjustified assumptions. Choosing the projection with maximum entropy is equivalent to a strategy of minimizing the maximum risk, which is proportional to our lack of knowledge of the true value of the quantity described by the fuzzy number.

Step 7. Output R.

In the proposed algorithm, the following operations are computationally complex:

- Calculation of the membership function $\omega_{ij} \in [0,1]$ of each element of the fuzzy binary relation D;
- Scanning of all elements of the fuzzy binary relation D to determine the maximums of the membership function and their indices (s_k, c_k);
- Calculating the uncertainty degree $f(E_l)$ of each potential solution E_l.

However, in practice, the speed of modern computers allows, without any tangible delay, to calculate a sufficiently large number of arithmetic expressions when solving applied problems.

3. Results

Example 1. *Summation of two NFSs with symmetric membership functions.*

Table 1 shows the two operands A and B as well as the fuzzy binary relation on Cartesian product $D = A + B$ in accordance with Equation (8) and two potential solutions (in bold). The result obtained based on the Zadeh's extension principle is located on the diagonal of the table.

Table 1. Two NFSs A and B, fuzzy binary relation $D = A + B$ and potential solutions E_1, E_2.

NFSs A										
1/0	1.2/0.2	1.4/0.4	1.6/0.6	1.8/0.8	2/1.0	2.2/0.8	2.4/0.6	2.6/0.4	2.8/0.2	3/0
NFSs B										
5/0	5.4/.02	5.8/0.4	6.2/0.6	6.6/0.8	7/1.0	7.4/0.8	7.8/0.6	8.2/0.4	8.6/0.2	9/0

Table 1. *Cont.*

Fuzzy Binary Relation on Cartesian Product $D = A + B$										
6/0	6.4/0	6.8/0	7.2/0	7.6/0	**8/0**	8.4/0	8.8/0	9.2/0	9.6/0	10/0
6.2/0	6.6/0.2	7/0.2	7.4/0.2	7.8/0.2	**8.2/0.2**	8.6/0.2	9/0.2	9.4/0.2	9.8/0.2	10.2/0.2
6.4/0	6.8/0.2	7.2/0.4	7.6/0.4	8/0.4	**8.4/0.4**	8.8/0.4	9.2/0.4	9.6/0.4	10/0.2	10.4/0
6.6/0	7/0.2	7.4/0.4	7.8/0.6	8.2/0.6	**8.6/0.6**	9/0.6	9.4/0.6	9.8/0.4	10.2/0.4	10.6/0
6.8/0	7.2/0.2	7.6/0.4	8/0.6	8.4/0.8	**8.8/0.8**	9.2/0.8	9.6/0.6	10/0.4	10.4/0.2	10.8/0
7/0	**7.4/0.2**	**7.8/0.4**	**8.2/0.6**	**8.6/0.8**	**9/1.0**	**9.4/0.8**	**9.8/0.6**	**10.2/0.4**	**10.6/0.2**	**11/0**
7.2/0	7.6/0.2	8/0.4	8.4/0.6	8.8/0.8	**9.2/0.8**	9.6/0.8	10/0.6	10.4/0.4	10.8/0.2	11.2/0
7.4/0	7.8/0.2	8.2/0.4	8.6/0.6	9/0.6	**9.4/0.6**	9.8/0.6	10.2/0.6	10.6/0.4	11/0.2	11.4/0
7.6/0	8/0.2	8.4/0.4	8.8/0.4	9.2/0.4	**9.6/0.4**	10/0.4	10.4/0.4	10.8/0.4	11.2/0.2	11.6/0
7.8/0	8.2/0.2	8.6/0.2	9/0.2	9.4/0.2	**9.8/0.2**	10.2/0.2	10.6/0.2	11/0.2	11.4/0.2	11.8/0
8/0	8.4/0	8.8/0	9.2/0	9.6/0	**10/0**	10.4/0	10.8/0	11.2/0	11.6/0	12/0

The relation D has only one element with a maximal degree of membership $d_{6,6} = (9, 1.0)$. The set E consists of two potential solutions: $E_1 = \{d_{6,j}, j = \overline{1,n}\}$ and $E_2 = \{d_{i,6}, i = \overline{1,n}\}$ which have uncertainty degrees $f(E_1) = 2,\ f(E_2) = 1$. In accordance with the proposed algorithm, the summation result $R = \{(z_{6j}, \omega_{6j}), j = \overline{1,n}\}$ is located in the row and has the support power $|\text{Supp}R| = 4$.

The result of standard fuzzy arithmetic.

The summation result Q, calculated according to the Zadeh's extension principle, is located on the diagonal of the fuzzy relation D. The uncertainty degree of this solution is $f(Q) = 3$, and the support power is $|\text{Supp}Q| = 6$.

As we can see, the proposed algorithm chose the operand with the maximum degree of uncertainty as a solution. However, the support of the resulting NFS is 33% smaller than the support of the result obtained using standard fuzzy arithmetic.

Example 2. *Summation of two NFSs with asymmetric membership functions.*

Table 2 shows the two operands A and B as well as two potential solutions to their summation. To be able to compare the result with known approaches, the NFSs are taken the same as the NFSs in **Figure 1**.

Table 2. The summation result of two operands A and B with asymmetric membership functions.

NFSs A										
1/0	1.4/0.2	1.8/0.4	2.2/0.6	2.6/0.8	3/0.9	3.4/1.0	3.8/0.7	4.2/0.4	4.6/0.2	5/0
NFSs B										
1/0	1.4/0.2	1.8/0.4	2.2/0.7	2.6/1.0	3/0.9	3.4/0.8	3.8/0.6	4.2/0.4	4.6/0.2	5/0
Summation Result E_1=A+B										
4.4/0	4.8/0.2	5.2/0.4	5.6/0.7	6/1.0	6.4/0.9	6.8/0.8	7.2/0.6	7.6/0.4	8/0.2	8.4/0
Summation Result E_2=A+B										
3.6/0	4/0.2	4.4/0.4	4.8/0.6	5.2/0.8	5.6/0.9	6/1.0	6.4/0.7	6.8/0.4	7.2/0.2	7.6/0

This example shows that in some cases the proposed algorithm requires additional conditions for choosing a solution R from E.

As we see, two NFSs from the set of potential solutions E have the same support power; their membership functions are asymmetric with respect to their own maximums, but symmetric with respect to each other. These two decisions have an equal degree of uncertainty: $f(E_1) = f(E_2) = 2.08$, that is, any of these solutions can be the result. What decision the algorithm chooses will depend only on its implementation in the software. The result obtained here can be interpreted as the ambiguity of the solution. However, it is entirely due to the symmetry of the input fuzzy numbers, which can be considered a logical "OR".

Note that the shape of the membership function in E_1 corresponds to operand B, and in E_2 to operand A. Therefore, the membership function of the result will repeat the membership function of one of the operands. If the operands have different support power, the membership function will take the shape of the operand, which has a great degree of uncertainty. This operand contains more information and absorbs operand with less uncertainty, which can be considered understandable from the viewpoint of the principle of maximum entropy. In other words, the proposed algorithm selects a solution with a maximal degree of uncertainty and at the same time seeks to preserve information about the shape of the membership function of the initial NFSs.

The result of standard fuzzy arithmetic.

From the viewpoint of the principle of maximum entropy, in contrast to the proposed algorithm, standard fuzzy arithmetic preserves the uncertainty of the original operands by increasing the support power, but loses information about the shape of the membership function of the operands. This can be seen in **Figure 3**, where the operand A is more flat on the right, the operand B is more flat on the left and more uncertain (since Card$B$ $>$ Card$A$). However, the result $(A+B)$ took the form of a fuzzy number with less uncertainty, which defies rational explanation.

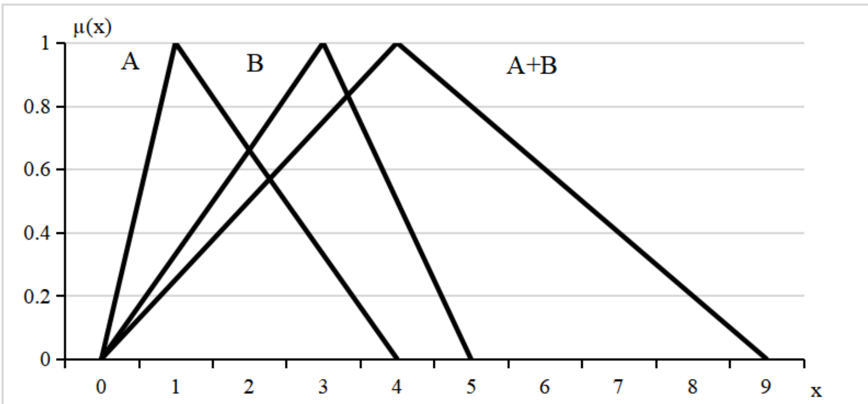


Figure 3. Changing the shape of the membership function when summing two triangular fuzzy numbers with different uncertainties.

The proposed algorithm forms a better solution based on the measurement of uncertainty.

Example 3. *Increasing the support of the resulting NFS in case of multiple arithmetic transformations.*

Let us consider a change in the support of the NFS as a result of executing several arithmetic operations on the discretized NFSs A and B shown in **Figure 2**. Discretization will not affect the consideration, since we are interested in the support, which is presented accurately. **Table 3** shows two NFSs and the summation result $(A+B+B+B+B+B)$.

Table 3. Two NFSs and the summation result $(A+B+B+B+B+B)$.

NFSs A										
1/0	1.4/0.14	1.8/0.29	2.2/0.43	2.6/0.57	3/0.71	3.4/0.86	3.8/1.0	4.2/0.66	4.6/0.33	5/0
NFSs B										
1/0	1.4/0.33	1.8/0.66	2.2/1.0	2.6/0.86	3/0.71	3.4/0.57	3.8/0.43	4.2/0.29	4.6/0.14	5/0
Summation Result $A+B$										
4.8/0	5.2/0.33	5.6/0.66	6/1.0	6.4/0.86	6.8/0.71	7.2/0.57	7.6/0.43	8/0.29	8.4/0.14	8.8/0
Summation Result $A+B+B$										
8.6/0	9/0.33	9.4/0.66	9.8/1.0	10.2/0.86	10.6/0.71	11/0.57	11.4/0.43	11.8/0.29	12.2/0.14	12.6/0
Summation Result $A+B+B+B$										
12.4/0	12.8/0.33	13.2/0.66	13.6/1.0	14/0.86	14.4/0.71	14.8/0.57	15.2/0.43	15.6/0.29	16/0.14	16.4/0
Summation Result $A+B+B+B+B$										
16.2/0	16.6/0.33	17/0.66	17.4/1.0	17.8/0.86	18.2/0.71	18.6/0.57	19/0.43	19.4/0.29	19.8/0.14	20.2/0
Summation Result $A+B+B+B+B+B$										
20/0	20.4/0.33	20.8/0.66	21.2/1.0	21.6/0.86	22/0.71	22.4/0.57	22.8/0.43	23.2/0.29	23.6/0.14	24/0

As we can see, the support power of the result does not change: $|\text{Supp}(A+B)| = |\text{Supp}(A+B+B)| = \ldots = 4$.

The result of standard fuzzy arithmetic.

As shown above (see **Figure 2**), the support of the result $|\text{Supp}(A+B+B+B+B)| = 24$ is much larger than in our algorithm.

In practice, the proposed algorithm allows us to obtain a result suitable for decision-making.

4. Discussion

4.1. Theoretical Questions

The use of the proposed algorithm is related to two questions that require discussion.

Question 1. On the Need to Observe the Properties of Standard Arithmetic

Judging by the number of publications, many authors consider it necessary to strictly observe the properties of standard arithmetic of real numbers when executing arithmetic operations with fuzzy numbers. However, as was shown above, the effectiveness of standard arithmetic in practice is doubtful due to a sharp increase in uncertainty. These doubts are the motive for revising the requirements for the properties of fuzzy arithmetic and raise a conceptual question: why should fuzzy arithmetic adhere to the rules that were introduced in standard arithmetic? Let's try to answer this question.

Fuzzy sets are an extension of traditional sets. In particular, the concept of fuzzy sets made it possible to use an extended range of values to measure the membership of elements in a set, and also made it possible to present standard sets as a special case of fuzzy sets.

But extension does not mean that the constraints of the original concept remain relevant. On the contrary, one can assume that a special case is produced from a general case by introducing additional constraints that narrow the original concept. Therefore, the desire to extend the properties of arithmetic of real numbers to fuzzy arithmetic is doubtful. This assumption raises the following question: how to change the properties of arithmetic of real numbers?

The answer should flow from the nature of uncertainty as a generalization of certainty. We can agree with the statement of the study regarding the replacement of the relation of strict equivalence by the relation of additive and multiplicative (that is, weak) equivalence[17]. Since fuzzy numerical quantities can be represented as a generalization of standard numerical quantities, it is logical to present the properties of arithmetic operations on fuzzy numbers as a generalization of the properties of standard arithmetic.

Question 2. About the Zadeh's Extension Principle

According to this principle, the result of an arithmetic operation is formed by the union of all elements' combinations of the support of both operands in order to prevent information loss. This combination is the reason for a sharp increase in the support power of the resulting NFS when calculating the arithmetic expression.

In the first numerical example, you can see that according to the Zadeh's extension principle, the result of the arithmetic operation is located on the diagonal of the relation D. However, this is true only for the case of symmetric membership functions of both operands. If we move the maximum of the membership function of one of the operands, then the result support will remain on the diagonal, but the maximum of the membership function will move to another place. That is, in the relation D, the resulting NFS will delaminate. Using the supremum operation avoids the influence of this effect, however, because of this, it is difficult to explain the change in the shape of the membership function.

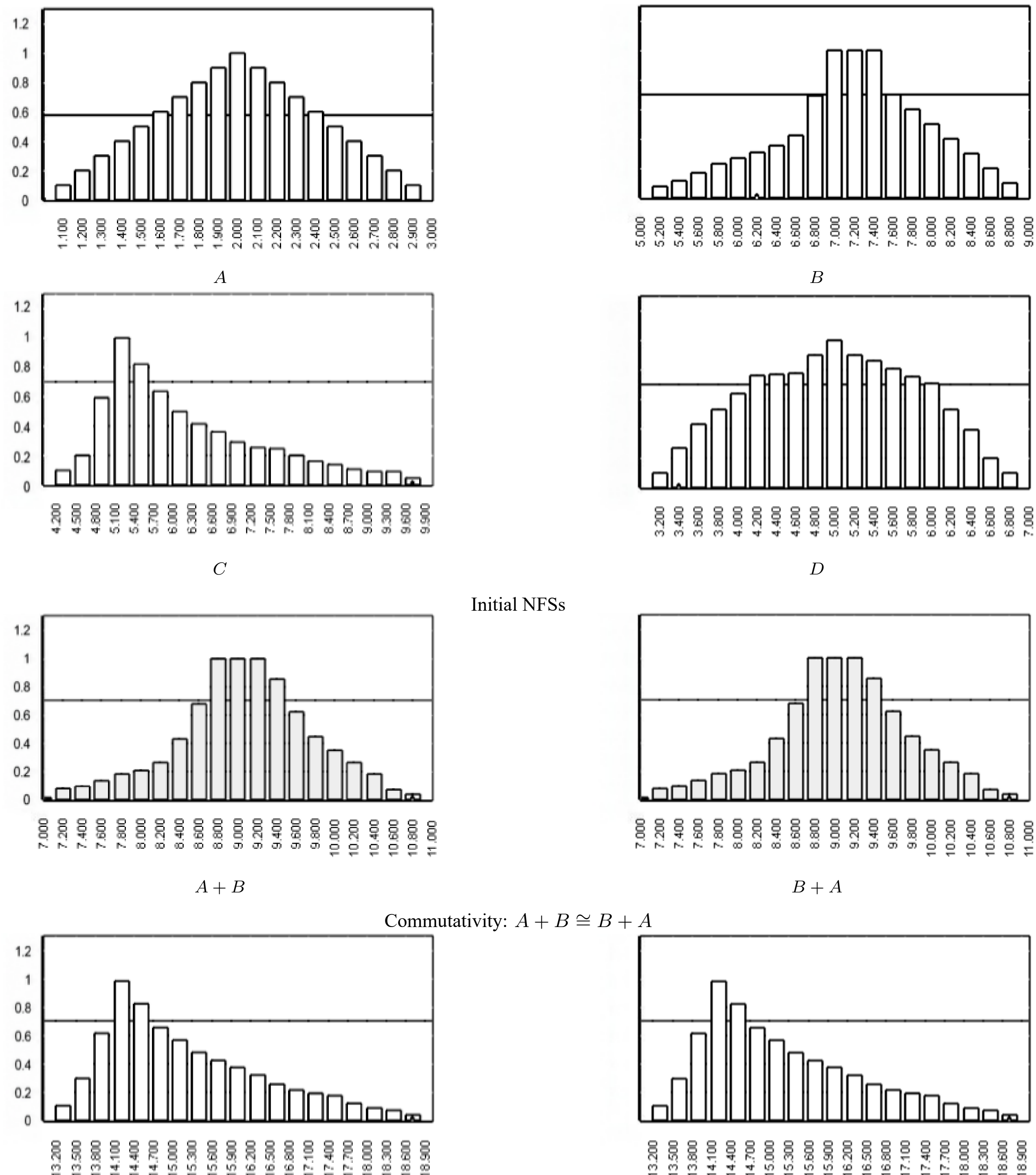
The result calculated on the basis of the Zadeh's extension principle is similar to the diagonal projection of a fuzzy binary relation, which is obtained by convolution using the supremum:

$$R = \{(r_j, \varphi_j), r_j \in \mathbb{R}, \varphi_j \in [0,1],\ j = \overline{1,n}, \}, \qquad (12)$$

where $r_j = z_{i=j}, i, j = \overline{1,n},\ \varphi_j = \sup\limits_{i=\overline{1,n}} \omega_{ij}$.

Then, the proposed algorithm can be considered as an algorithm for calculating a conditional projection of the same relation D, whereas the condition is the maximum of the membership Equation (9). In other words, the proposed algorithm obtains a solution not by convolution of the elements of the relation, but by choosing one of the conditional projections.

Returning to question 1 discussed above, the following can be noted. Since the proposed algorithm uses an additional condition that narrows the generalized solution, the solutions of this algorithm will be more constrained, but preserve the properties of the generalized concept. **Figure 4** demonstrates the preservation of the properties of traditional fuzzy arithmetic in the proposed algorithm. Screenshots were obtained using the Fuzzy for Excel add-in.



A B

C D

Initial NFSs

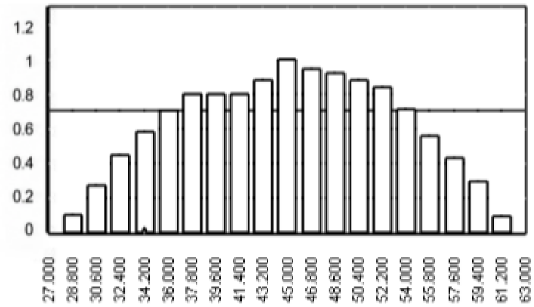
$A+B$ $B+A$

Commutativity: $A+B \cong B+A$

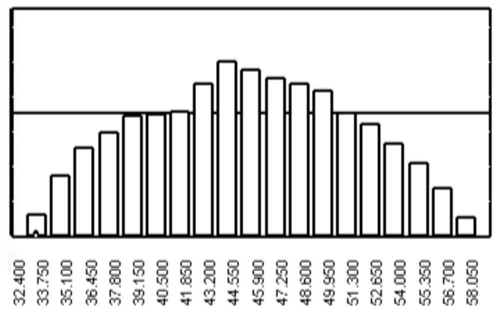
$(A+B)+C$ $A+(B+C)$

Associativity: $(A+B)\ +\ C \cong A\ +\ (B+C)$

Figure 4. *Cont.*



$D \cdot (A + B)$



$(D \cdot A) + (D \cdot B)$

Distributivity: $D \cdot (A + B) \cong (D \cdot A)\ +\ (D{\cdot}B)$

Figure 4. Preserving the properties of traditional fuzzy arithmetic in the proposed algorithm.

4.2. Comparison with Other Fuzzy Arithmetic Methods

Comparison with other methods provides a basis for a more in-depth analysis of the proposed method. Therefore, we compared our method with two known methods. The first method is the Zadeh's extension principle (see Equation (3)). The overwhelming majority of known methods are based on it. Although the Zadeh's extension principle does not reduce the support power of the resulting NFS, it represents the most complete set of solutions for the arithmetic operation. The second method is the relatively recently published QKB method[28], which is designed only for multiplying triangular numbers. Despite this, we see no obstacles to its use in other types of arithmetic operations. As for other methods, we did not find any studies that would be aimed at reducing the support power.

For comparison, we use the criterion of the support power of the resulting NFS in arithmetic operations with various triangular NFSs whose support is defined on different parts of the real line. For simplicity, we use two operations: addition and multiplication. Subtraction and division are inverse operations and therefore may be ignored. Triangular NFSs are described here by a tuple (l, m, n), similar to the work[25]: l, m and n are lower-pointed, main-pointed and upper-pointed values respectively. Taking into account the problems of fuzzy arithmetic (see above), we chose three parts of the real number set: the negative region, the zero-crossing region, and the positive region. In the zero-crossing region, we also investigated various cases of mutual arrangement of fuzzy numbers. All these combinations cover most of the possible combinations of NFSs in practice.

For the calculation by our method, we used the software described below. For the calculation according to Zadeh'smethod (see Equation (3)), we used the approach described by Dubois and Prade[25]. For the calculation by the QKB-method, we used the expressions described by Al-Qudaimi et al[28]. The comparison results are presented in **Table 4**.

Table 4. Results of methods comparison.

Operation and Operands (l, m, n)	Resulting NFSs (l, m, n)	Support Power $\lvert l - m \rvert$
Zadeh's Extension Principle		
$(-20, -18, -16) + (-36, -33, -30)$	$(-56, -51, -46)$	10
$(-20, -18, -16) + (-5, -1, +3)$	$(-25, -19, -13)$	12
$(-20, -18, -16) + (+30, +33, +36)$	$(+10, +15, +20)$	10
$(-5, +1, +3) + (-20, -18, -16)$	$(-25, -17, -13)$	12
$(-5, +1, +3) + (-10, -2, +6)$	$(-5, -1, +9)$	14
$(-5, +1, +3) + (-2, +2, +6)$	$(-7, +3, +9)$	16
$(-5, +1, +3) + (-10, -4, +2)$	$(-15, -3 + 5)$	20
$(-5, +1, +3) + (+30, +33, +36)$	$(+25, +34, +39)$	14
$(+30, +33, +36) + (-20, -18, -16)$	$(+10, +15, +20)$	10
$(+30, +33, +36) + (-5, -1, +3)$	$(+25, +32, +39)$	14
$(+30, +33, +36) + (+30, +33, +36)$	$(+60, +66, +72)$	12
$(-20, -18, -16) * (-36, -33, -30)$	$(+720, +594, +480)$	240
$(-20, -18, -16) * (-5, -1, +3)$	$(+100, +18, -48)$	no solution

Table 4. *Cont.*

Operation and Operands (l, m, n)	Resulting NFSs (l, m, n)	Support Power $\|l - m\|$
Zadeh's Extension Principle		
$(-20, -18, -16) * (+30, +33, +36)$	$(-600, -594, -576)$	24
$(-5, +1, +3) * (-20, -18, -16)$	$(+100, -18, -48)$	no solution
$(-5, +1, +3) * (-10, -2, +6)$	$(+50, -2, +18)$	no solution
$(-5, +1, +3) * (-2, +2, +6)$	$(+10, +2, +18)$	no solution
$(-5, +1, +3) * (-10, -4, +2)$	$(+50, -4, +6)$	no solution
$(-5, +1, +3) * (+30, +33, +36)$	$(-150, +33, +108)$	258
$(+30, +33, +36) * (-20, -18, -16)$	$(-600, -594, -576)$	24
$(+30, +33, +36) * (-5, -1, +3)$	$(-150, -33, +108)$	258
$(+30, +33, +36) * (+30, +33, +36)$	$(+900, +1, 089, +1, 296)$	396
QKB-Method		
$(-20, -18, -16) + (-36, -33, -30)$	$(-56, -51, -46)$	10
$(-20, -18, -16) + (-5, -1, +3)$	$(-25, -19, -13)$	12
$(-20, -18, -16) + (+30, +33, +36)$	$(+10 + 15, +20)$	10
$(-5, +1, +3) + (-20, -18, -16)$	$(-25, -17, -12)$	13
$(-5, +1, +3) + (-10, -2, +6)$	$(-13, -1, +9)$	22
$(-5, +1, +3) + (-2, +2, +6)$	$(-9, +3, +13)$	22
$(-5, +1, +3) + (-10, -4, +2)$	$(-15, -3, +5)$	20
$(-5, +1, +3) + (+30, +33, +36)$	$(+26, +34, +39)$	15
$(+30, +33, +36) + (-20, -18, -16)$	$(+10, +15, +20)$	10
$(+30, +33, +36) + (-5, -1, +3)$	$(+25, +32, +39)$	14
$(+30, +33, +36) + (+30, +33, +36)$	$(+60, +66, +72)$	12
$(-20, -18, -16) * (-36, -33, -30)$	$(+588, +594, +600)$	12
$(-20, -18, -16) * (-5, -1, +3)$	$(+10, +18, +26)$	16
$(-20, -18, -16) * (+30, +33, +36)$	$(-600, -594, -588)$	12
$(-5, +1, +3) * (-20, -18, -16)$	$(-26, -18, -14)$	12
$(-5, +1, +3) * (-10, -2, +6)$	$(-46, -2, +14)$	60
$(-5, +1, +3) * (-2, +2, +6)$	$(-22, +2, +10)$	32
$(-5, +1, +3) * (-10, -4, +2)$	$(-40, -4, +8)$	48
$(-5, +1, +3) * (+30, +33, +36)$	$(+15, +33, +39)$	24
$(+30, +33, +36) * (-20, -18, -16)$	$(-600, -594, -588)$	12
$(+30, +33, +36) * (-5, -1, +3)$	$(-45, -33, -21)$	24
$(+30, +33, +36) * (+30, +33, +36)$	$(+1, 080, +1, 089, 1, 098)$	18
Proposed Method		
$(-20, -18, -16) + (-36, -33, -30)$	$(-54, -51, -48)$	6
$(-20, -18, -16) + (-5, -1, +3)$	$(-23, -19, -15)$	8
$(-20, -18, -16) + (+30, +33, +36)$	$(+12, +15, +18)$	6
$(-5, +1, +3) + (-20, -18, -16)$	$(-19, -17, -15)$	4
$(-5, +1, +3) + (-10, -2, +6)$	$(-9, -1, +7)$	16
$(-5, +1, +3) + (-2, +2, +6)$	$(-1, +3, +7)$	8
$(-5, +1, +3) + (-10, -4, +2)$	$(-9, -3, +3)$	12
$(-5, +1, +3) + (+30, +33, +36)$	$(+31, +34, +37)$	6
$(+30, +33, +36) + (-20, -18, -16)$	$(+13, +15, +17)$	4
$(+30, +33, +36) + (-5, -1, +3)$	$(+28, +32, +36)$	8
$(+30, +33, +36) + (+30, +33, +36)$	$(+63, +66, +69)$	6
$(-20, -18, -16) * (-36, -33, -30)$	$(+528, +594, +660)$	132
$(-20, -18, -16) * (-5, -1, +3)$	$(-54, +18, +90)$	144
$(-20, -18, -16) * (+30, +33, +36)$	$(-660, -594, -528)$	132
$(-5, +1, +3) * (-20, -18, -16)$	$(-54, -18, +90)$	144
$(-5, +1, +3) * (-10, -2, +6)$	$(-10, -2, +6)$	16
$(-5, +1, +3) * (-2, +2, +6)$	$(-2, +2, +6)$	8
$(-5, +1, +3) * (-10, -4, +2)$	$(-12, -4, +20)$	32
$(-5, +1, +3) * (+30, +33, +36)$	$(-165, +33, +99)$	164
$(+30, +33, +36) * (-20, -18, -16)$	$(-660, -594, -528)$	132
$(+30, +33, +36) * (-5, -1, +3)$	$(-165, -33, +99)$	164
$(+30, +33, +36) * (+30, +33, +36)$	$(+990, +1, 089, +1, 188)$	198

The analysis of the results shows that the Equation (3) may not provide solutions if the support of one of the NFSs contains a zero-crossing. Compared with other methods, the support power of the resulting NFS is maximal. The QKB method in the multiplication operation is the best method according to the criterion of the support power of the resulting NFS. However, in the summation operation, the support power is greater than in the proposed algorithm. In addition, the QKB method cannot be used for non-triangular NFSs. Overall, the most problematic case is arithmetic operations with NFSs whose carrier contains a zero-crossing. In such cases, the proposed method increases the support spread, but provides a solution. The QKB method requires additional research regarding possible information losses.

4.3. Software "Fuzzy for Excel"

Many researchers did not consider the developed algorithms only as scientific advances. They implemented these algorithms with the help of special software packages, such as "Mathematica", and used them to solve applied problems [29–32]. Sometimes researchers themselves developed special programs [33–35]. However, these programs have not become widespread due to the narrow specialization or complexity of the user interface. In other words, the problem of the availability of fuzzy arithmetic for a wide range of engineers and analysts remains relevant.

Therefore, the authors developed the "Fuzzy for Excel" add-in for Microsoft Excel office software, which is widely used to solve applied problems in various fields of science, technology and business. This add-in can be downloaded for free from the link: https://1drv.ms/f/c/6a215c39cc7b74ec/Eq_Y17O-kVtOoOb5EmLyDecB9lRLlvWE3yUXyaJ0Kw03LA.

Installation of the add-in does not require special knowledge and can be performed in accordance with the guide.

To simplify, in the software, numerical fuzzy sets are called fuzzy numbers. They are represented in the form of discretized fuzzy sets, which have 21 pairs (support value / membership function value). The add-in implements many tools and functions that provide the creation, processing and analysis of numerical fuzzy sets. The add-in also contains several functions for describing and processing uncertainty using fuzzy Sugeno measures and fuzzy integrals [36]. **Table 5** shows the functions that are implemented in the add-in.

Table 5. Functions that are implemented in the "Fuzzy for Excel" add-in.

Function	Purpose
FuzzyAverage	calculates the average of several fuzzy numbers
FuzzyComplementation	inverts the membership function
FuzzyConvolution	computes the convolution of several fuzzy numbers taking into account weighting factors
FuzzyCopy	copies a fuzzy number from one cell of a worksheet to another
FuzzyDegreeFunction	raises the membership function to a given power
FuzzyEquivalenceSets	returns the measure of similarity between two sets
FuzzyFigure	specifies a number as a geometric figure
FuzzyFormula	calculates the result of arithmetic operations
FuzzyGetBearer	returns the value of the support of a fuzzy number according to the discrete number
FuzzyGetCell	copies a fuzzy number from a fuzzy memory cell to a worksheet cell
FuzzyGetDiscount	discounts a fuzzy number
FuzzyGetLevel	returns the confidence value of the number's support
FuzzyGetMembership	returns the confidence value of the support of a number according to the discrete number
FuzzyGetNumberForRisk	returns a number that corresponds to a given risk level
FuzzyGetNumbers	returns a pair of ordinary numbers that match the given confidence level
FuzzyGetParameter	returns the value of the given fuzzy number parameter
FuzzyGetRiskLessThan	returns the risk that a fuzzy number will be less than a given number
FuzzyGetRiskMoreThan	returns the risk that a fuzzy number will be greater than a given number
FuzzyHand	specifies a number of arbitrary shape according to the parameters specified as strings
FuzzyHandFromCells	specifies a number of arbitrary shape in accordance with parameters that are specified as cell ranges
FuzzyIntegral	returns the value of the fuzzy integral of a fuzzy number over a fuzzy measure that is described by the fuzzy number
FuzzyIntegralSemantic	returns the value of the fuzzy integral of a membership function with respect to a fuzzy measure on a crisp set
FuzzyIntegralSemanticFS	returns the value of the fuzzy integral of a membership function with respect to a fuzzy measure on a fuzzy set
FuzzyIntegralSemanticExtFS	returns the value of the extended fuzzy integral of a membership function over a fuzzy measure that is induced in another space
FuzzyGetLambdaParameter	returns the normalization parameter of a fuzzy measure
FuzzyIntersection	returns the coordinates of the intersection of two fuzzy lines
FuzzyInterval	specifies a fuzzy number as an interval
FuzzyLessThan	specifies a fuzzy number with semantics "less than ..."

Table 5. *Cont.*

Function	Purpose
FuzzyLessThanTo	specifies a fuzzy number with semantics "less than ... to ..."
FuzzyMakeFromStatistic	specifies a number based on a statistical sample
FuzzyMax	calculates the union of several numbers
FuzzyMin	calculates the intersection of several numbers
FuzzyMoreThan	specifies a fuzzy number with semantics "greater than ..."
FuzzyMoreThanTo	specifies a fuzzy number with semantics "greater than ... to ..."
FuzzyNear	specifies a fuzzy number with semantics "about ..."
FuzzyNearAndLessTo	specifies a fuzzy number with semantics "about ... and less than up to ..."
FuzzyNearAndMoreTo	specifies a fuzzy number with semantics "about ... and more up to ..."
FuzzyNearFirstOrSecond	specifies a fuzzy number with semantics "about (... or ...)"
FuzzyNearFromTo	specifies a number of the type "about (from ... to ...)"
FuzzyProduct	calculates the product of several numbers
FuzzySetCell	copies a fuzzy number from a worksheet cell to a fuzzy memory cell
FuzzySum	calculates the sum of several numbers

The basics of using the add-in, as well as descriptions of tools and functions, are contained in the help system.

4.4. Example of Calculating the Profit and Currency Risk of Options

An option is a financial contract. This contract requires the buyer to pay a premium for the rights provided by the contract. A call option allows you to buy an asset (such as a currency) at a specified price within a specified period of time. A put option allows you to sell an asset at a specified price within a specified period of time.

Exercising an option means exercising the right to buy or sell an asset at a specified price. When buying/selling an option, the most important factor is the option strike price, which should be based on the forecast of the future price of the asset. Using an option allows the owner to avoid significant losses in the event of sharp changes in the asset price. With minor fluctuations in this price, the owner may refuse to exercise the option. In this case, he will lose the premium paid, which is essentially a fee for risk insurance. The profit and risk of the option seller are directly opposite to the profit and risk of the option owner. The maximum profit of the option seller is equal to the premium if the buyer refuses to exercise the option.

Thus, when buying/selling an option, it is necessary to forecast the change in the asset price. When concluding a contract, this allows you to estimate the expected profit and possible risk.

Let us consider the US dollar as an asset and calculate the profit and risks when selling an option using the above-described software "Fuzzy for Excel". Note that the forecast of the key parameter—the UAH/USD exchange rate—is presented as a triangular fuzzy number, since fuzzy numbers describe the uncertainty of the future well.

Statement of the problem: Let a bank sell a put option to some enterprise to buy US dollars (USD) for hryvnia (UAH). The problem is to assess the risk of no profit when exercising the option. The terms of the option sale are as follows:

- Option cost (OC)—5,000 thousand USD;
- Strike price (SP)—42.2561 UAH/USD;
- Option premium (OP)—2% from revenue (R);
- Forecast exchange rate (FER) at the time of option strike—from 42.5 to 46 UAH/USD, the most expected value is 44.351 UAH/USD.

Solution to the problem: The forecast rate at the time of option exercise cannot be precisely determined. However, it can be described by a fuzzy numerical value, which is shown in **Figure 5**.

To solve the problem, we use an algorithm that sequentially calculates the following indicators:

- Revenue (R) from option sales at the rate: $R = OC{\cdot}SP$;
- Option premium: $OP = R \cdot 0.02$;
- Net income (NI) with taking into account of premium: $NI = R - OP = R \cdot 0.98$;
- Real selling rate (RSR): $RSR = NI/OC$;
- Forecast revenue (FR) with taking into account of forecast exchange rate: $FR = OC \cdot FER$;
- Profit (P): $P = (FER - SP) \cdot OC - OP$;
- Forecast net revenues (FNR): $FNR = FR - OP$.

The results of the calculation of the put option indicators are shown in **Table 6**.

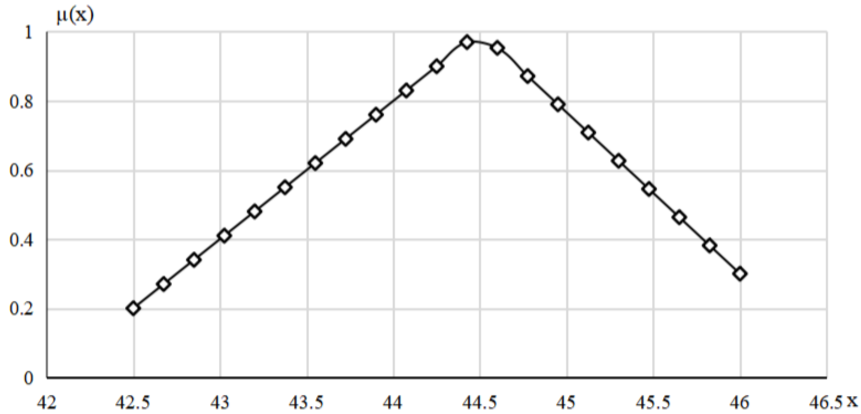


Figure 5. Fuzzy numerical value of FER at the time of option exercise.

Table 6. Results of calculating put option indicators.

Indicator	Value	Unit of Measurement
option cost (OC)	5,000	thousand USD
strike price (SP)	42.2561	UAH/USD
option premium (OP)	4,225.61	thousand UAH
forecast exchange rate (FER)	**44.351**	**UAH/USD**
revenue from option sales (R)	211,280.5	thousand UAH
net income (NI)	207,054.89	thousand UAH
real selling rate (RSR)	41.4110	UAH/USD
forecast revenue (FR)	**222,125**	**thousand UAH**
profit (P)	**6,618.89**	**thousand UAH**
forecast net revenues (FNR)	**217,899.39**	**thousand UAH**

Note: Bold font designates fuzzy numerical values.

Calculations show that net proceeds after premium payment will be 222,125 thousand UAH. This is equivalent to the option being exercised at the rate of 41.4110 UAH/USD. However, the forecast UAH/USD rate is determined by the fuzzy value shown in **Figure 5**. In this case, net income NI may be 207,054.89 thousand UAH (the most expected value). The NI value is described by the fuzzy number shown in **Figure 6**.

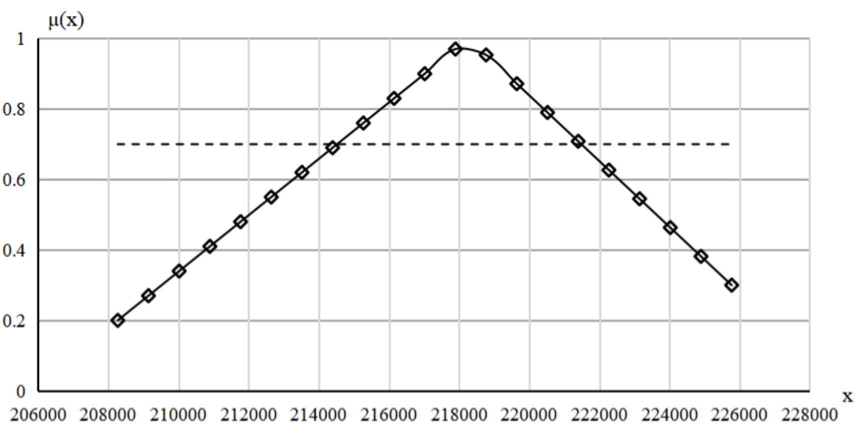


Figure 6. The NI value.

At the confidence level of 0.7, the expected spread of values is from 215,274.39 to 221,399.39 thousand UAH. It can be expected that as a result of purchasing the option, the enterprise can receive a profit, the most expected value of which is about 8,184 thousand UAH. The spread of forecast profit FR is shown in **Figure 7**.

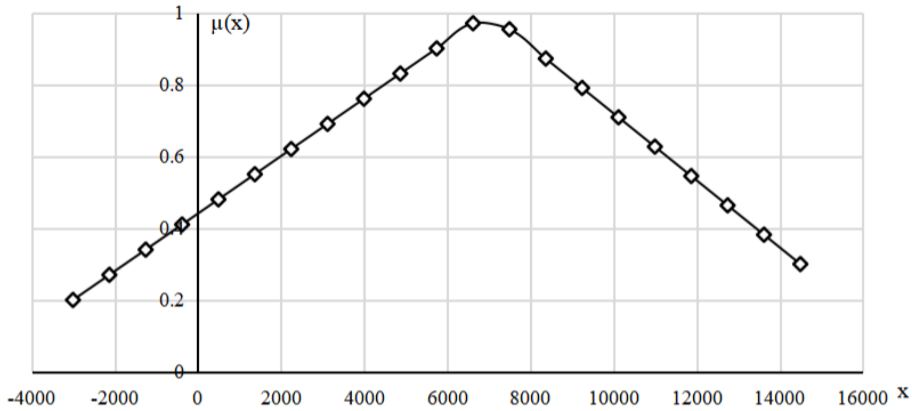


Figure 7. The FR value.

The maximum possible profit from the exercise of the option on a confidence level of 0.5 (50%) is 11,868.89 thousand UAH.

The results of calculations using fuzzy arithmetic also allow us to assess risks. As can be seen from **Figure 7**, in the case of an unfavorable exchange rate forecast, the enterprise will receive a loss, the maximum value of which may be 3,006.11 thousand UAH. The possibility of this loss is 0.2 (20%). **Figure 8** shows the risk function of profit [24].

Looking at **Figure 8**, we can conclude that the risk of loss when executing the option is approximately −0.53 or 53%.

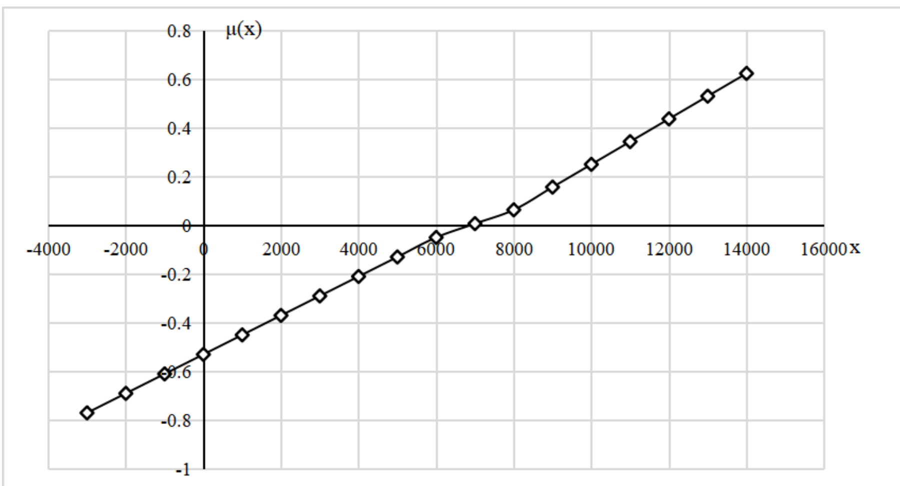


Figure 8. Risk function of profit P.

The proposed algorithm has been used to develop many projects, including:

- Actuarial calculations for determining third-party liability insurance rates;
- Profit calculations for several infrastructure solution options during the construction of an innovation park;
- Calculating the efficiency of factoring operations;
- Profit forecasting for the construction of a city's multi-service information and telecommunications infrastructure and others.

The calculations considered above show the possibility of using fuzzy arithmetic to solve applied problems in business. In addition, the proposed algorithm was used to assess risks in an agricultural project [37].

5. Conclusions

The proposed algorithm for executing arithmetic operations preserves the uncertainty nature of operands and does not contradict the basic properties of standard fuzzy arithmetic. The algorithm limits the increase in the support of the resulting NFS, which allows to improve the data quality for decision-making in applied problems. This is especially important when executing a large number of arithmetic operations. The proposed algorithm has great computational complexity, but this is not an obstacle to the implementation of the algorithm in software, given the speed of modern computers.

The algorithm is implemented as an add-in to a well-known office software. To use the add-in, users do not need any special knowledge. However, users must have skills in working with uncertain numerical values.

Patents

The State Department of Intellectual Property has issued a software "Fuzzy for Excel" patent #26511 to the authors.

Author Contributions

Conceptualization, S.S. and V.B.; methodology, V.B.; software, S.S.; validation, S.S.; formal analysis, V.B.; investigation, S.S.; resources, S.S.; data curation, S.S.; writing—original draft preparation, S.S.; writing—review and editing, V.B.; visualization, S.S.; supervision, V.B.; project administration, S.S. Both authors have read and agreed to the published version of the manuscript.

Funding

This work received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

The software implementing the research results is freely available at the following link: https://1drv.ms/f/c/6a215c39cc7b74ec/Eq_Y17O-kVtOoOb5EmLyDecB9lRLlvWE3yUXyaJ0Kw03LA.

Conflicts of Interest

The authors declare no conflict of interest.

AI Use Statement

The authors declare that no artificial intelligence (AI) tools were used in the preparation of this manuscript.

References

[1] Zadeh, L.A., 1975. The concept of a linguistic variable and its application to approximate reasoning—II. Information Sciences. 8(4), 301–357. DOI: https://doi.org/10.1016/0020-0255(75)90046-8

[2] Dubois, D., Prade, H., 1978. Operation on fuzzy numbers. International Journal of Systems Science. 9(6), 613–626. DOI: https://doi.org/10.1080/00207727808941724

[3] Yager, R.R., 1980. On the lack of inverses in fuzzy arithmetic. Fuzzy Sets and Systems. 4(1), 73–83. DOI: https://doi.org/10.1016/0165-0114(80)90065-2

[4] Mizumoto, M., Tanaka, K., 1976. The four operations on fuzzy numbers. Systems, Computers, Controls. 7(5), 73–81.

[5] Klir, G.J., 1997. Fuzzy arithmetic with requisite constraints. Fuzzy Sets and Systems. 91(2), 165–175. DOI: https://doi.org/10.1016/S0165-0114(97)00138-3

[6] Hesamian, G., Johannssen, A., 2024. On arithmetic operations of LR fuzzy numbers with different shape functions. IEEE Transactions on Fuzzy Systems. 32(11), 6583–6587. DOI: https://doi.org/10.1109/TFUZZ.2024.3447081

[7] Gáspár, S., Musinszki, Z., Hágen, I.Z., et al., 2023. Developing a controlling model for analyzing the subjectivity of enterprise sustainability and expert group judgments using fuzzy triangular membership functions. Sustainability. 15(10), 7981. DOI: https://doi.org/10.3390/su15107981

[8] Bede, B., 2006. Product type operations between fuzzy numbers and their applications in geology. Acta Polytechnica Hungarica. 3(1), 123–139.

[9] Kané, L., Diakité, M., Kané, S., et al., 2021. A new algorithm for fuzzy transportation problems with trapezoidal fuzzy numbers under fuzzy circumstances. Journal of Fuzzy Extension and Applications. 2(3), 204–225. DOI: https://doi.org/10.22105/jfea.2021.287198.1148

[10] Srinivasan, R., Karthikeyan, N., Jayaraja, A., 2021. A proposed ranking method to solve transportation prob-

lem by pentagonal fuzzy numbers. Turkish Online Journal of Qualitative Inquiry. 12(3), 277–286. Available from: https://surl.li/myiwtp

[11] Lv, H., Wang, G., Li, C., 2023. Approximations of n-dimensional fuzzy numbers by using α −β-knots piecewise linear fuzzy n-cell numbers. Fuzzy Sets and Systems. 467, 108494. DOI: https://doi.org/10.1016/j.fss.2023.02.015

[12] Appadoo, S.S., Gajpal, Y., Kumar, A., 2021. An application of possibilistic moments of nonlinear type of fuzzy numbers in supply chain management. Mathematical Problems in Engineering. 2021(1), 5538185. DOI: https://doi.org/10.1155/2021/5538185

[13] Sharm, R., Pal, A., 2023. An improved analysis on approximation of fuzzy numbers. AIP Conference Proceedings. 2495(1), 020066. DOI: https://doi.org/10.1063/5.0123355

[14] Velu, L.G.N., Ramalingam, B., 2023. Total ordering on generalized 'n'gonal linear fuzzy numbers. International Journal of Computational Intelligence Systems. 16(1), 23. Available from: https://link.springer.com/content/pdf/10.1007/s44196-022-00180-8.pdf

[15] Hanss, M., 1999. On the implementation of fuzzy arithmetical operations for engineering problems. In Proceedings of the 18th International Conference of the North American Fuzzy Information Processing Society (NAFIPS), New York, NY, USA, 10–12 June 1999; pp. 462–466. DOI: https://doi.org/10.1109/NAFIPS.1999.781736

[16] Piegat, A., Landowski, M., 2022. Are multidimensional RDM interval arithmetic and constrained interval arithmetic one and the same? Iranian Journal of Fuzzy Systems. 19(5), 17–34. Available from: https://ijfs.usb.ac.ir/article_7154_bd8bd56fb0d26f7372decca1fce0e65d.pdf

[17] Mizukoshi, M.T., Lodwick, W.A., 2022. Interval arithmetic: WSM, CI or RDM? In: Rayz, J., Raskin, V., Dick, S., et al. (Eds.). Explainable AI and Other Applications of Fuzzy Techniques. Springer: Cham, Switzerland. pp. 291–301.

[18] Leelavathi, R., et al., 2025. Analyzing solutions of fuzzy nabla dynamic equations on timescales: A generalized Hukuhara approach. Reliability: Theory & Applications. 20(3), 769–783. Available from: https://www.gnedenko.net/Journal/2025/032025/RTA_3_2025-63.pdf

[19] Zabihi, S., Ezzati, R., Fattahzadeh, F., et al., 2024. Fuzzy fractional calculus: A comprehensive overview with a focus on weighted Caputo-type generalized Hukuhara differentiability and analytical solutions for fuzzy fractional differential equations. Iranian Journal of Fuzzy Systems. 21(2), 19–34.

[20] Mareš, M., 1997. Weak arithmetics of fuzzy numbers. Fuzzy Sets and Systems. 91(2), 143–153. DOI: https://doi.org/10.1016/S0165-0114(97)00136-X

[21] Piasecki, K., Łyczkowska-Hanćkowiak, A., 2021. Oriented fuzzy numbers vs. fuzzy numbers. Mathematics. 9(5), 523. DOI: https://doi.org/10.3390/math9050523

[22] Stefanini, L., Laerte, S., Guerra, M.L., 2008. Fuzzy Numbers and Fuzzy Arithmetic. In: Pedrycz, W., Skowron, A., Kreinovich, V. (Eds.). Handbook of Granular Computing. John Wiley & Sons: Hoboken, NJ, USA. DOI: https://doi.org/10.1002/9780470724163.ch12

[23] Liu, H., Guo, S., 2007. Equality and Identity of Fuzzy Numbers and Fuzzy Arithmetic with Equality Constraints. In Proceedings of the 2007 International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2007), Chengdu, China, 15–16 October 2007; pp. 334–339.

[24] Bocharnikov, V., 2001. Fuzzy-Technology: Mathematical Foundations. Modeling Practice in Economics. Nauka RAN: Saint Petersburg, Russia. (in Russian)

[25] Dubois, D., Prade, H., 1980. Fuzzy Sets and Systems: Theory and Application. Academic Press: New York, NY, USA.

[26] Kotelnikov, V.A., 1933. On the Transmission Capacity of "Ether" and Wire in Electro-Communications. Available from: https://ict.open.ac.uk/classics/assets/documents/1.pdf (cited 19 August 2025).

[27] Dhar, M., 2013. On Some Properties of Entropy of Fuzzy Numbers. International Journal of Intelligent Systems and Applications. 5(3), 66–73. DOI: https://doi.org/10.5815/ijisa.2013.03.07

[28] Al-Qudaimi, A., Kaur, K., Bhat, S., 2021. Triangular fuzzy numbers multiplication: QKB method. Fuzzy Optimization and Modeling Journal. 2(2), 34–40.

[29] Chen, S.-M., 1996. Evaluating weapon systems using fuzzy arithmetic operations. Fuzzy Sets and Systems. 77(3), 265–276. DOI: https://doi.org/10.1016/0165-0114(95)00096-8

[30] Kosheleva, O., Kreinovich, V., Bouchon-Meunier, B., et al., 1998. Operations with Fuzzy Numbers Explain Heuristic Methods in Image Processing. Available from: https://scholarworks.utep.edu/cgi/viewcontent.cgi?article=1548&context=cs_techrep (cited 19 August 2025).

[31] Zimmermann, H.-J., Zysno, P., 1983. Decisions and evaluations by hierarchical aggregation of information. Fuzzy Sets and Systems. 10(1–3), 243–260. DOI: https://doi.org/10.1016/S0165-0114(83)80118-3

[32] Kochegurova, E.A., Martynova, Y.A., 2020. Aspects of Continuous User Identification Based on Free Texts and Hidden Monitoring. Programming and Computer Software. 46, 12–24. DOI: https://doi.org/10.1134/S036176882001003X

[33] Koleśnik, R., Prokopowicz, P., Kosiński, W., 2004. Fuzzy Calculator—Useful Tool for Programming with Fuzzy Algebra. In: Rutkowski, L., Siekmann, J.H., Tadeusiewicz, R., et al. (Eds.). Artificial Intelligence and Soft Computing (ICAISC 2004). Springer:

Berlin/Heidelberg, Germany. pp. 320–325. DOI: https://doi.org/10.1007/978-3-540-24844-6_45

[34] Gagolewski, M., Caha, J., 2018. A Guide to the FuzzyNumbers Package for R (FuzzyNumbers version 0.4-7). Available from: https://vps.fmvz.usp.br/CRAN/web/packages/FuzzyNumbers/vignettes/FuzzyNumbersTutorial.pdf (cited 19 August 2025).

[35] Marin, M., Defour, D., Milano, F., 2016. An Efficient Representation Format for Fuzzy Intervals Based on Symmetric Membership Functions. ACM Transactions on Mathematical Software. 43(3), 1–22. DOI: https://doi.org/10.1145/2939364

[36] Sugeno, M., 1974. Theory of fuzzy integrals and its applications [PhD Thesis]. Tokyo Institute of Technology: Tokyo, Japan.

[37] Sveshnikov, S., Bocharnikov, V., Uvarova, T., et al., 2024. Algorithms for assessing the qualitative and quantitative risks of lending to small and medium-sized businesses based on fuzzy calculus. Serbian Journal of Management. 19(1), 1–31.